A bridge between Schrödinger equation and Schrödinger bridge process

Léon Brenig (leon.brenig@ulb.be)
Marc Vincke (marc.vincke@ulb.be)

Université Libre de Bruxelles

May 2025

Unitary (deterministic) time evolution

Generalities

- Scope : quantum non-relativistic spinless particle subject to an external scalar potential or to an external electromagnetic field

$$H = \frac{1}{2m}(-i\hbar\nabla - qA(x))^2 - qE(x)\cdot x + V(x) \tag{1}$$

- Unitary evolution of the wave function $\rightarrow$ Schrödinger equations

$$i\hbar\partial_t\psi(x,t) = +H\psi(x,t) \tag{2}$$

$$i\hbar\partial_t\psi^*(x,t) = -H^*\psi^*(x,t) \tag{3}$$

- with $t \in [0, T = 1]$ and initial condition $\psi(x, t = 0)$
- Born rule providing the probability in QM $\rightarrow$ probability density

$$\rho(x,t) = \psi^*(x,t)\psi(x,t) \tag{4}$$

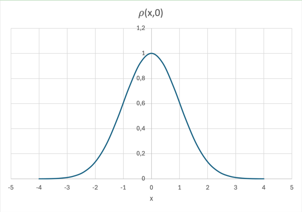
Unitary (deterministic) time evolution

Example 1 - 1D wave packet - Quantitative

- Initial condition (choice of units) :
  - Rest inertial frame $< x >= 0$, $< p >= 0$
  - Variances $< \Delta x^2 >= \sigma$, $< H >=< \Delta p^2 > /2 = (1 + \theta^2)/8\sigma$

$$\psi(x,0) = \exp(-\frac{(1+i\theta)x^2}{4\sigma}) \tag{5}$$

$$\rho(x,0) = \exp(-\frac{x^2}{2\sigma}) \tag{6}$$



Unitary (deterministic) time evolution

Example 1 - 1D wave packet - Quantitative

- Solution of Schrödinger equation (2) for $t \in [0,1]$ with initial condition (5)

$$\psi(x,t) = \exp\left(-\frac{(1+i\Theta(t,\theta))x^2}{4\sigma a(t,\theta)}\right), \tag{7}$$

$$\rho(x,t) = \exp\left(-\frac{x^2}{2\sigma a(t,\theta)}\right), \tag{8}$$

introducing the relative width $a(t,\theta) > 1$ and the phase parameter $\Theta(t,\theta)$, for $t > 0$

$$a(t,\theta) = (1-\theta t/2\sigma)^2 + (t/2\sigma)^2 \tag{9}$$

$$\Theta(t,\theta) = \theta - (1+\theta^2)t/2\sigma \tag{10}$$

Unitary (deterministic) time evolution

Example 1 - 1D wave packet - Quantitative

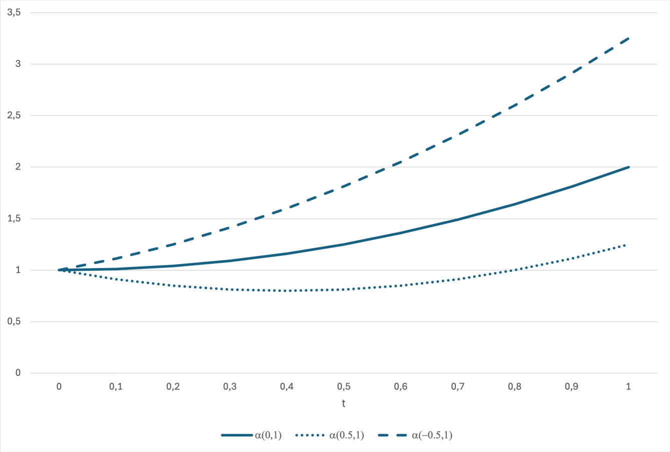
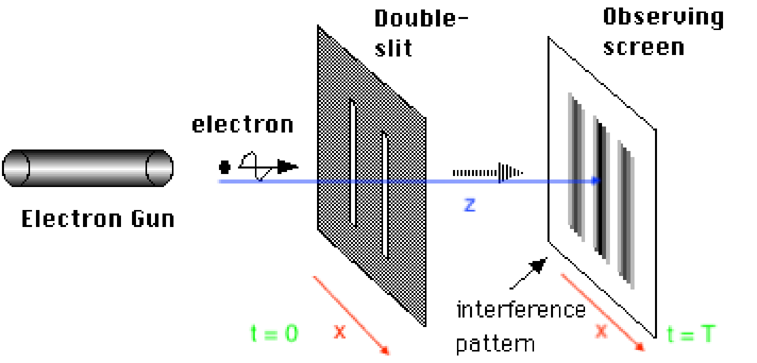


FIGURE 1 – Relative widths $a(t,\theta)$ for the numerical examples $\sigma = 1/2$ and $\theta = -0.5, 0, +0.5$.

Unitary (deterministic) time evolution

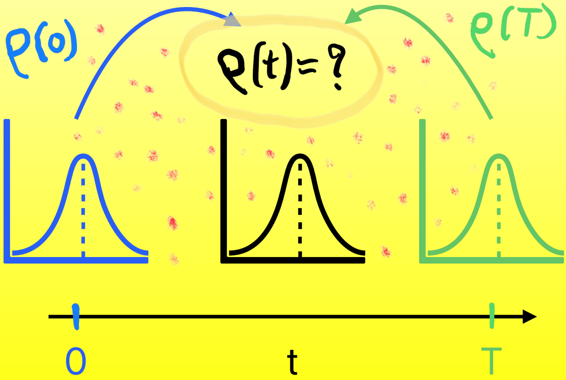
Example 2 - Two-slits experiment - Qualitative



Schrödinger bridge

Schrödinger's idea

Schrödinger [Schrödinger, 1931] [Schrödinger, 1932] tried to find a classical probabilistic derivation with characteristics are as close as possible to those of the Schrödinger equation (2)



Schrödinger bridge

Schrödinger's idea

- Resulting equations $\rightarrow$ real positive functions φ, ϕ

$$\hbar\partial_\tau\varphi(x,\tau) = -H\varphi(x,\tau) \quad (11)$$
$$\hbar\partial_\tau\phi(x,\tau) = +H^*\phi(x,\tau) \quad (12)$$

- Similar to heat equations $\rightarrow$ irreversible phenomena like stochastic diffusion or Bernstein processes [Zambrini, 1986] [Zambrini, 1987], which are as different as possible from quantum phenomena
- To solve this contradiction : by imposing the same boundary conditions at the time interval bounds for :
  - the Schrödinger-bridge Born rule $\varphi\phi$,
  - the usual quantum-mechanical Born rule $\psi^*\psi$.

Schrödinger bridge

Example 1 : 1D wave packet

- Resolution of the Schrödinger-bridge equations (11, 12) $\rightarrow$ resulting density (the analog of (8))

$$\rho(x,\tau) = \varphi(x,\tau)\phi(x,\tau) = \exp(-\frac{x^2}{2\sigma A(\tau,\alpha)}) \quad (13)$$

- Introducing the reduced Schrödinger-bridge width

$$A(\tau,\alpha) = (1+\frac{1}{\alpha}\frac{\tau}{\sigma})(1-(1-\frac{1}{\alpha})\frac{\tau}{\sigma}) \quad (14)$$

- The α parameter is obtained by imposing the equivalence of Born-rule boundary conditions $A(1,\alpha) = a(1,\theta)$

Schrödinger bridge

Example 1 : 1D wave packet

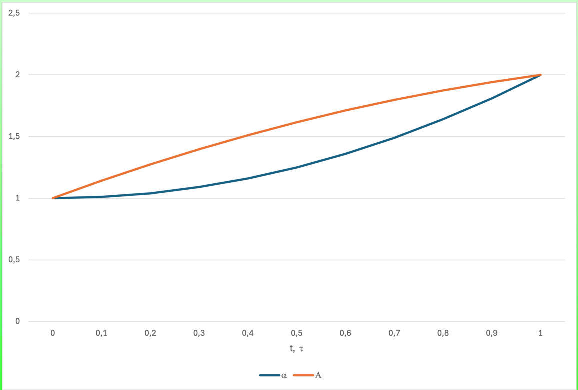


FIGURE 2 – Comparison of the unitary and Schrödinger-bridge relative widths $a(t,0)$ and $A(\tau,\alpha)$ for the numerical example $\sigma = 1/2$.

Schrödinger bridge

Schrödinger's idea

- These two articles of Schrödinger went almost totally unknown to physicists. However, not to mathematicians in stochastic optimal control and stochastic theory !
- At the end of his article [Schrödinger, 1931], E. Schrödinger concluded :

  > *(But) I do not wish to analyse these points more closely before time tells if they can really lead to a better understanding of quantum mechanics.*

- And so he did ! He never published another article on this subject. . .
- **However...**

Schrödinger bridge

What's next ?

- Come-back of Schrödinger's idea in QM from the 80s [Zambrini, 1986] [Zambrini, 1987]
  - Euclidean quantum mechanics (the closest classical analogy of quantum mechanics)
  - Scope extended (free particle $\rightarrow$ external scalar potential added)
  - The solutions φ, ϕ of the Schrödinger system (11, 12) exist and are unique for a given inital density and any final density
- Two questions arise
  - How are the unitary Schrödinger-equation and non-unitary Schrödinger-bridge dynamics connected ?
  - Is the Schrödinger-bridge evolution parameter τ identical to the usual time and if not, what is it and how is it connected to usual time ?

$\mathcal{S}$ transformations and related axioms

Note : premises of this part of the presentation to be found in [Brenig, 2007]

Hamilton-Jacobi description of classical mechanics : Free classical non relativistic particle of mass m

$$\partial_t S = -\frac{\mid \nabla S \mid^2}{2m} \tag{15}$$

where $S(x)$ is the action of the system.

Random initial position x of probability density $\rho(x)$: Continuity equation

$$\partial_t \rho = -\nabla.(\rho \frac{\nabla S}{m}) \tag{16}$$

$\mathcal{S}$ transformations and related axioms

Average of the position and momentum

$$< x >= \int d^3x\rho x \; ; < p >= \int d^3x\rho\nabla S$$

Choice of Rest Inertial Frame

$$< x >= 0 \; ; < p >= 0$$

Classical variances of x and p

$$Var(x) = \int d^3x\rho x^2 \; ; \; Var(p) = \int d^3x\rho \mid \nabla S \mid^2$$

In the rest frame the classical average kinetic energy is $Var(p)/2m$

$\mathcal{S}$ transformations and related axioms

UNIVERSITÉ LIBRE DE BRUXELLES

Quantum description

For polar form of wave function $\psi \equiv \rho^{1/2} e^{iS/\hbar}$ the Schrödinger equation becomes [Bohm and Hiley, 1984] :

$$\partial_t S = -\frac{|\nabla S|^2}{2m} + \frac{\hbar^2}{2m}\frac{\triangle \rho^{1/2}}{\rho^{1/2}} \tag{17}$$

$$\partial_t \rho = -\nabla.(\rho \frac{\nabla S}{m})$$

Question :

Where is the extra term $\frac{\hbar^2}{2m}\frac{\triangle \rho^{1/2}}{\rho^{1/2}}$ in equation (17) from ?

$\mathcal{S}$ transformations and related axioms

Nonlinear gauge transformations (NLGT) acting on ρ and S :

$$\rho(\alpha) = \rho\,;\, S(\alpha) = e^{-\alpha} S\,;\, \alpha \in \mathbb{R} \tag{18}$$

Here ρ is **normalised to one**. Obviously, the NLGT are nonlinear transformations on ψ

Claim : Quantum extra-term in equation (17) emerges from an invariance postulate under the NLGT (18)

$\mathcal{S}$ transformations and related axioms

ULB UNIVERSITÉ LIBRE DE BRUXELLES

Postulate 1 : Invariance Postulate :

The Heisenberg uncertainty principle must remain invariant under the nonlinear gauge transformations.

However : Product $Var(x).Var(p)$ does not fulfill the Invariance Postulate ! !

Demonstration :

Suppose

$$Var(x).Var(p) \geq \frac{\hbar^2}{4}$$

Apply NLGT :

$$Var(x,\alpha).Var(p,\alpha) = e^{-2\alpha} Var(x).Var(p)$$

$$Lim_{\alpha \to +\infty} Var(x,\alpha).Var(p,\alpha) = 0$$

$\mathcal{S}$ transformations and related axioms

How to fulfill Postulate 1 ?

Replace $Var(x)$ and $Var(p)$ respectively by :

$$Var(x) \Rightarrow \triangle_x^2 \equiv \frac{1}{4\int d^3x \mid \nabla\rho^{1/2} \mid^2} \tag{19}$$

$$Var(p) \Rightarrow \triangle_p^2 \equiv Var(p) + \frac{\hbar^2}{4}\frac{1}{\triangle_x^2} \tag{20}$$

$\triangle_x^2$ **is the Fisher length and satisfies the Cramér-Rao inequality [Cox and Hinkley, 1979] :**

$$Var(x) \geq \triangle_x^2 \tag{21}$$

ULB UNIVERSITÉ LIBRE DE BRUXELLES

How to fulfill Postulate 1 ? Sequel.

Then, apply NLGT to product $\triangle_x^2 \triangle_p^2$:

$$\triangle_x^2(\alpha)\triangle_p^2(\alpha) = e^{-2\alpha}\triangle_x^2\triangle_p^2 + (1 - e^{-2\alpha})\frac{\hbar^2}{4} \quad (22)$$

Taking limit $\alpha \to +\infty$ in (22) :

$$Lim_{\alpha \to +\infty}\triangle_x^2(\alpha)\triangle_p^2(\alpha) = \frac{\hbar^2}{4} \quad (23)$$

$\mathcal{S}$ transformations and related axioms

How to fulfill Postulate 1 ? Sequel.

But : $Var(x, \alpha) = Var(x)$ **and** $\triangle_x^2(\alpha) = \triangle_x^2$

Apply Cramér-Rao inequality (21) to equation (23)

$$Lim_{\alpha \to +\infty} Var(x, \alpha) \triangle_p^2(\alpha) \geq \frac{\hbar^2}{4} \tag{24}$$

Consequently, $Var(x)$ **and** $\triangle_p^2$ **are the correct measures of uncertainties, respectively on** x **and** p**, that fulfill the Invariance Postulate**

$\mathcal{S}$ transformations and related axioms

Postulate 2 :

In the rest frame the average kinetic energy is given by $\triangle_p^2/2m$

This postulate generalises $Var(p)/2m$ **Average Kinetic Energy, in rest frame**

Check that Postulate 1 and Postulate 2 imply quantum mechanics :

Postulate 2 + Postulate 1 (19, 20) $\Longrightarrow$ **Average Kinetic Energy :**

$$\mathcal{H}_0 \equiv \triangle_p^2/2m = \int d^3x[\rho \frac{|\nabla S|^2}{2m} + \frac{\hbar^2}{2m} |\nabla \rho^{1/2}|^2] \qquad (25)$$

$\mathcal{S}$ transformations and related axioms

Retrieving quantum mechanics

$\mathcal{H}_0$ belongs to Lie algebra $\mathbb{G}$ of functionals $\mathcal{A}$ of ρ and S [Guerra and Marra, 1983] :

$$\mathcal{A} = \int \mathrm{d}^3\mathrm{x} F(x, \rho, \nabla\rho, \nabla\nabla\rho, \ldots, S, \nabla S, \nabla\nabla S, \ldots)$$

with functional Poisson Lie brackets :

$$\{\mathcal{A}, \mathcal{B}\} = \int \mathrm{d}^3\mathrm{x}\left[\frac{\delta\mathcal{A}}{\delta\rho(\mathrm{x})}\frac{\delta\mathcal{B}}{\delta S(x)} - \frac{\delta\mathcal{B}}{\delta\rho(x)}\frac{\delta\mathcal{A}}{\delta S(x)}\right]$$

Technical remark : To take account of $\int d^3x\rho(x) = 1$, one must replace $\rho(x)$ by $\frac{\rho(x)}{\int d^3x\rho(x)}$ when performing functional derivatives !

$\mathcal{S}$ transformations and related axioms

ULB UNIVERSITÉ LIBRE DE BRUXELLES

Time evolution :

$$\partial_t \mathcal{A} = \{\mathcal{A}, \mathcal{H}_0\} \tag{26}$$

$\mathcal{H}_0$ is infinitesimal generator of one-parameter Lie group of parameter $t \in \mathbb{R}$: $e^{t\{.,\mathcal{H}_0\}}$

Equation (26) applied to ρ and S retrieves Equations (16, 17)

$\Longrightarrow$ retrieves Schrödinger equation for ψ !

Note : With $\psi \equiv \rho^{1/2} e^{iS/\hbar}$ one obtains :

$$\mathcal{H}_0 = \int d^3x \psi^* (-i\hbar\nabla)^2 \psi$$

$\Longrightarrow$ $\mathcal{H}_0$ belongs to sub-algebra $\mathbb{Q}$ of $\mathbb{G}$, of quantum averages of Hermitian operators

$\mathcal{S}$ transformations and related axioms

ULB UNIVERSITÉ LIBRE DE BRUXELLES

What did we get up to now ?

- **Imposing Postulates 1 and 2 on classical mechanics in Hamilton-Jacobi version lead to Schrödinger equation**
- ***Invariance of* Heisenberg inequality under NLGT is analogue to *invariance of* special relativity inequality $v \leq c$ under Lorenz transformations. *Planck constant $\hbar$ analogue to light velocity c !***
- **But there is more ! ! ! Postulates 1 and 2 generate also another dynamics**

$\mathcal{S}$ transformations and related axioms

NLGT are generated by functional $\mathcal{S} \in \mathbb{G}$:

$$\mathcal{S} = \int d^3 x \rho S \tag{27}$$

For any functional $\mathcal{A}$ in $\mathbb{G}$:

$$\frac{\partial \mathcal{A}}{\partial \alpha} = \{\mathcal{A}, \mathcal{S}\}$$

$\mathcal{S}$ generates a one-parameter Lie group with parameter $\alpha \in \mathbb{C}$:
$e^{\alpha\{.,\mathcal{S}\}}$

$$\mathcal{A}(\alpha) = e^{\alpha\{.,\mathcal{S}\}} \mathcal{A} \tag{28}$$

Equations (27, 28) applied to ρ and S give :

$$\rho \to \rho(\alpha) = \rho \, ; \, S \to S(\alpha) = e^{-\alpha} S$$

$\Longrightarrow$ identical to NLGT equation (18) !

Thus, $\mathcal{S}$-transformations equivalent to NLGT...

$\mathcal{S}$ transformations and related axioms

UNIVERSITÉ LIBRE DE BRUXELLES

How another dynamics emerges from the Postulates ?

Start with Equation (22) :

$$\triangle_x^2(\alpha)\triangle_p^2(\alpha) = e^{-2\alpha}\triangle_x^2\triangle_p^2 + (1 - e^{-2\alpha})\frac{\hbar^2}{4}$$

Use $\triangle_x^2(\alpha) = \triangle_x^2$ and divide Equation (22) by $\triangle_x^2(\alpha)$ + Use expressions (19, 20) of $\triangle_x^2$ and $\triangle_p^2$ leads to :

$$e^{\alpha}\mathcal{H}_0(\alpha) = [\cosh(\alpha)\mathcal{H}_0 - \sinh(\alpha)\mathcal{K}_0] \qquad (29)$$

where

$$\mathcal{K}_0 = \int d^3x[\rho\frac{|\nabla S|^2}{2m} - \frac{\hbar^2}{2m}|\nabla\rho^{1/2}|^2] \qquad (30)$$

$\mathcal{S}$ transformations and related axioms

How another dynamics emerges from the Postulates ? Sequel.

$\Longrightarrow$ **A new functional of $\mathbb{G}$, $\mathcal{K}_0$**

$\Longrightarrow$ **Generates a one-parameter Lie group with parameter $\tau \in \mathbb{R}$:** $e^{\tau\{.,\mathcal{K}_0\}}$

A new dynamics emerges : a dynamics in another time-like parameter τ

$\mathcal{S}$-transformation (NLGT) of $\mathcal{K}_0$:

$$e^{\alpha}\mathcal{K}_0(\alpha) = [-\sinh(\alpha)\mathcal{H}_0 + \cosh(\alpha)\mathcal{K}_0] \quad (31)$$

$\Longrightarrow$ ***Hyperbolic rotation*** **of the couple $(H_{0,}\mathcal{K}_0)$ followed by dilatation :**

$$\mathcal{H}_0(\alpha) = e^{-\alpha}[\cosh(\alpha)\mathcal{H}_0 - \sinh(\alpha)\mathcal{K}_0] \quad (32)$$

$$\mathcal{K}_0(\alpha) = e^{-\alpha}[-\sinh(\alpha)\mathcal{H}_0 + \cosh(\alpha)\mathcal{K}_0] \quad (33)$$

ULB UNIVERSITÉ LIBRE DE BRUXELLES

$\mathcal{S}$ transformations of $\mathcal{H}$

ULB UNIVERSITÉ LIBRE DE BRUXELLES

$\mathcal{H}$ generates the time evolution of any functional $\mathcal{A}$ of $\mathbb{G}$

- Average-energy functional involving the Hamiltonian operator H (1)

$$\mathcal{H} = \int d^3x \psi^*(x) H \psi(x) \tag{34}$$

- Canonical transformation $\rightarrow$ canonical pair ψ, ψ^*

$$\{\mathcal{A}, \mathcal{B}\} = \frac{1}{i\hbar} \int d^3x [\frac{\delta \mathcal{A}}{\delta \psi(x)} \frac{\delta \mathcal{B}}{\delta \psi^*(x)} - \frac{\delta \mathcal{B}}{\delta \psi(x)} \frac{\delta \mathcal{A}}{\delta \psi^*(x)}] \tag{35}$$

- $\mathcal{H}$ generates evolution in $t \in \mathbb{R}$ of any functional $\mathcal{A}$ according to

$$\partial_t \mathcal{A} = \{\mathcal{A}, \mathcal{H}\} \tag{36}$$

- Particular case : the Schrödinger equation (2)

$\mathcal{S}$ transformations of $\mathcal{H}$

Obtained from the $\mathcal{S}$ transformations of ψ and ψ^*

- $\mathcal{S}$ transformations of ψ and ψ^* $(\alpha \in \mathbb{C})$

$$\psi(\alpha) = \psi^{\frac{1+e^{-\alpha}}{2}} (\psi^*)^{\frac{1-e^{-\alpha}}{2}} \tag{37}$$

$$\psi^*(\alpha) = \psi^{\frac{1-e^{-\alpha}}{2}} (\psi^*)^{\frac{1+e^{-\alpha}}{2}} \tag{38}$$

- Conserve the Born rule

$$\rho(\alpha) = \psi^*(\alpha)\psi(\alpha) = \psi^*\psi = \rho \tag{39}$$

- Then

$$\begin{aligned} \mathcal{H}(\alpha) &= \int d^3x \psi^*(\alpha) H \psi(\alpha) \\ &= \int d^3x \psi^{\frac{1-e^{-\alpha}}{2}} (\psi^*)^{\frac{1+e^{-\alpha}}{2}} H \psi^{\frac{1+e^{-\alpha}}{2}} (\psi^*)^{\frac{1-e^{-\alpha}}{2}} \end{aligned} \tag{40}$$

$\mathcal{S}$ transformations of $\mathcal{H}$

Connection with the Schrödinger bridge

- Taking $\alpha = \alpha_0 = -i\pi/2$
- Defining the real positive functions φ, ϕ from (37, 38)

$$\psi(\alpha_0) = \varphi \tag{41}$$
$$\psi^*(\alpha_0) = \phi \tag{42}$$

- $\mathcal{S}$ transformations of $\mathcal{H}$, from (40, 41, 42)

$$\mathcal{H}(\alpha_0) = i\mathcal{K} \tag{43}$$

- Introducing the average-energy-like functional

$$\mathcal{K} = -\int d^3x \phi H \varphi \tag{44}$$

$\mathcal{S}$ transformations of $\mathcal{H}$

Connection with the Schrödinger bridge

- Canonical transformation $\rightarrow$ canonical pair φ, ϕ

$$\{\mathcal{A}, \mathcal{B}\} = \frac{1}{\hbar} \int d^3x \left[\frac{\delta \mathcal{A}}{\delta \varphi(x)} \frac{\delta \mathcal{B}}{\delta \phi(x)} - \frac{\delta \mathcal{B}}{\delta \varphi(x)} \frac{\delta \mathcal{A}}{\delta \phi(x)} \right] \tag{45}$$

- $\mathcal{K}$ generates evolution in $\tau \in \mathbb{R}$ of any functional $\mathcal{A}$ according to

$$\partial_\tau \mathcal{A} = \{\mathcal{A}, \mathcal{K}\} \tag{46}$$

- Particular cases : the Schrödinger-bridge equations (11, 12)

$\mathcal{S}$ transformations of $\mathcal{H}$ and $\mathcal{K}$

Considering first a particular case : no magnetic field

- No magnetic field $\rightarrow$ from the definition of the Hamiltonian (1) :

$$H^* = H \tag{47}$$

- Under this condition, for any α, it can be shown that

$$\mathcal{H}(\alpha) = e^{-\alpha}[+\cosh(\alpha)\mathcal{H} - \sinh(\alpha)\mathcal{K}] \tag{48}$$

$$\mathcal{K}(\alpha) = e^{-\alpha}[-\sinh(\alpha)\mathcal{H} + \cosh(\alpha)\mathcal{K}] \tag{49}$$

- **The couple of the Schrödinger-equation dynamics and the Schrödinger-bridge dynamics is invariant under $\mathcal{S}$ transformations (NLGT)**
- $\mathcal{H}(\alpha)$ and $\mathcal{K}(\alpha)$ are the generators of two one-dimensional Lie groups with real parameters $t(\alpha)$ and $\tau(\alpha)$
- This answers the first question on slide 13

$\mathcal{S}$ transformations of $\mathcal{H}$ and $\mathcal{K}$

ULB UNIVERSITÉ LIBRE DE BRUXELLES

Considering first a particular case : no magnetic field

E. Schrödinger considered as a failure his attempt in 1931 to build a stochastic model of quantum mechanics explaining the origin of the Born rule : $\rho = \psi\psi^*$

Here, we showed that his model not only derives from the same postulates as standard quantum mechanics, but also it is intertwined with standard QM by the nonlinear gauge transformations

The Born rule appears naturally in the Schrödinger Bridge stochastic process. Its intertwining with standard QM extends the Born rule to standard QM

E. Schrödinger missed the point but was very near it !

$\mathcal{S}$ transformations of $\mathcal{H}$ and $\mathcal{K}$

Time reversal

- Time-reversed wave function

$$\psi_R(x,t) = \Theta\psi(x,-t) = \psi^*(x,-t) \tag{50}$$

- Time-reversal operator Θ [Littlejohn, 2021]

$$\Theta\psi = \psi^* \tag{51}$$

$$\Theta^\dagger\psi^* = \psi \tag{52}$$

$$\Theta^\dagger\Theta = \Theta\Theta^\dagger = 1 \tag{53}$$

$$\Theta^\dagger z\Theta = \Theta z\Theta^\dagger = z^*(\forall z \in \mathbb{C}) \tag{54}$$

- Time-reversed Schrödinger equation

$$i\hbar\partial_t\psi_R(x,t) = +\Theta H\Theta^\dagger\psi_R(x,t) \tag{55}$$

$\mathcal{S}$ transformations of $\mathcal{H}$ and $\mathcal{K}$

The general case $H^* \neq H$

- Taking $\alpha = 2\alpha_0 = -i\pi$ in (37, 38)

$$\psi(2\alpha_0) = \psi^* = \Theta\psi \tag{56}$$

$$\psi^*(2\alpha_0) = \psi = \Theta^\dagger\psi^* \tag{57}$$

- The time-reversal operator is a $\mathcal{S}$ transformation !
- $\mathcal{S}$ transformations of $\mathcal{H}$, from (40, 56, 57)

$$\mathcal{H}(2\alpha_0) = \mathcal{H}_R \tag{58}$$

- Introducing the time-reversed average-energy functional

$$\mathcal{H}_R = \int d^3x\psi H\psi^* = \int d^3x\psi^*\Theta^\dagger H\Theta\psi = \int d^3x\psi^* H^*\psi \tag{59}$$

$\mathcal{S}$ transformations of $\mathcal{H}$ and $\mathcal{K}$

The general case $H^* \neq H$

- $\mathcal{H}_R$ generates the evolution in t (the same parameter as for $\mathcal{H}$) of any functional $\mathcal{A}$ according to

$$\partial_t \mathcal{A} = \{\mathcal{A}, \mathcal{H}_R\} \tag{60}$$

- Key point : time evolution of the time-reversed system
- Particular case : the time-reversed Schrödinger equation (55)
- Same approach for the τ-reversed average-energy-like functional $\mathcal{K}_R$
- Θ is both the t and τ reversal operator

$\mathcal{S}$ transformations of $\mathcal{H}$, $\mathcal{K}$, $\mathcal{H}_R$ and $\mathcal{K}_R$

The general case $H^* \neq H$

- Much more complicated than the hyperbolic rotation - dilatation (48, 49)
- It can be shown that

$$\begin{pmatrix} \mathcal{H}(\alpha) \\ \mathcal{K}(\alpha) \\ \mathcal{H}_R(\alpha) \\ \mathcal{K}_R(\alpha) \end{pmatrix} = \begin{pmatrix} M_+(\alpha) & M_-(\alpha) \\ M_-(\alpha) & M_+(\alpha) \end{pmatrix} \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \\ \mathcal{H}_R \\ \mathcal{K}_R \end{pmatrix} \tag{61}$$

- Introducing

$$M_+(\alpha) = e^{\pm\frac{\alpha}{2}-\alpha} \begin{pmatrix} \cosh(\frac{\alpha}{2})\cosh(\alpha) & \mp i\cosh(\frac{\alpha-\alpha_0}{2})\sinh(\alpha) \\ \pm i\cosh(\frac{\alpha+\alpha_0}{2})\sinh(\alpha) & \cosh(\frac{\alpha}{2})\cosh(\alpha) \end{pmatrix} \tag{62}$$

$$M_-(\alpha) = \mp e^{\pm\frac{\alpha}{2}-\alpha} \begin{pmatrix} \sinh(\frac{\alpha}{2})\cosh(\alpha) & -i\cosh(\frac{\alpha+\alpha_0}{2})\sinh(\alpha) \\ +i\cosh(\frac{\alpha-\alpha_0}{2})\sinh(\alpha) & \sinh(\frac{\alpha}{2})\cosh(\alpha) \end{pmatrix} \tag{63}$$

ULB UNIVERSITÉ LIBRE DE BRUXELLES

What is τ ?

The transformations of average-energy and average-energy-like functionals, both without (48, 49) or with (61) magnetic field induce the same transformations of parameters t and τ :

$$t(\alpha) = e^{\alpha}[\cosh(\alpha)t + \sinh(\alpha)\tau] \tag{64}$$

$$\tau(\alpha) = e^{\alpha}[\sinh(\alpha)t + \cosh(\alpha)\tau] \tag{65}$$

(a hyperbolic rotation followed by a dilatation)

The two parameters t and τ are a priori independant. They are parameters of two different one-parameter Lie groups.

This answers the second question on slide 13

What is τ ?

How can we interpret τ ? One is tempted to call it a new dimension !

Schrödinger Bridge process is an evolution in this new dimension.

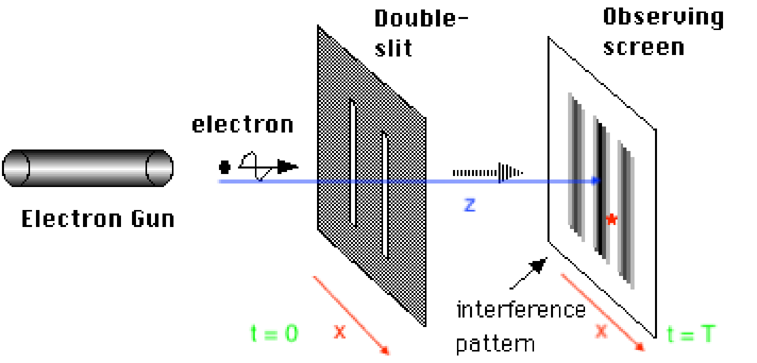
What does it represent ?
It could represent the measurement process. The Schrödinger Bridge requires two boundary conditions : initial and final conditions on a finite τ interval $[0, T]$.

***The final condition is compatible with the definition of a measurement process :* the measument apparatus is designed to induce a collapse of the wavefunction of the system on an eigenvector of the measured observable.**

Thus, the measurement interaction introduces a final condition !

What is τ ? Back to Schrödinger bridge

Example 2 - Two-slits experiment - Position measurement



What is τ ? Back to Schrödinger bridge

Example 1 : 1D wave packet - Position measurement ?

- At $\tau = 1$: accurate position measurement with $< x >= x(1)$ and $< \Delta x^2 >= \Sigma(1) \to 0$:

$$\rho(x,1) = \exp(-\frac{(x - x(1))^2}{2\Sigma(1)}) \tag{66}$$

- Resolution of the Schrödinger-bridge equations (11, 12) $\to$ resulting density

$$\rho(x,\tau) = \exp(-\frac{(x - \tau x(1))^2}{2\Sigma(\tau)}), \tag{67}$$

- introducing the Schrödinger-bridge width

$$\Sigma(\tau) = (\sigma + (1-\sigma)\tau)(1-\tau) \tag{68}$$

- The centre of the transformed density (67) evolves linearly from 0 to $x(1)$ with $\tau \in [0,1]$

What is τ ? Back to Schrödinger bridge

ULB UNIVERSITÉ LIBRE DE BRUXELLES

Example 1 : 1D wave packet - Position measurement ?

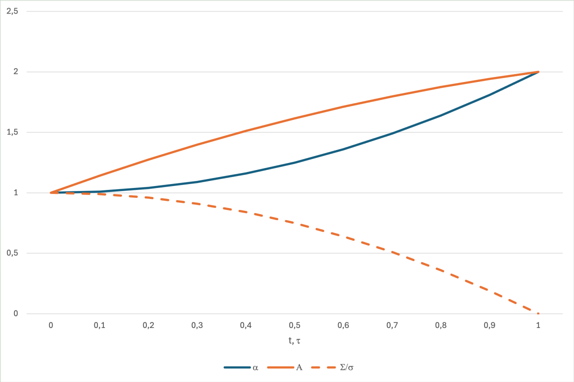


FIGURE 3 – Comparison of position-measurement relative width $\Sigma(\tau)/\sigma$ with the unitary and Schrödinger-bridge ones $a(t,0)$ and $A(\tau,\alpha)$, for the numerical example $\sigma = 1/2$.

Références I

Bohm, D. and Hiley, B. J. (1984).
Measurement understood through the quantum potential approach.
Found Phys, 14(3) :255–274.

Brenig, L. (2007).
Is quantum mechanics based on an invariance principle ?
Journal of Physics A : Mathematical and Theoretical, 40(17) :4567.

Cox, D. and Hinkley, D. (1979).
Theoretical Statistics.
Chapman and Hall/CRC, 0 edition.

Guerra, F. and Marra, R. (1983).
Origin of the quantum observable operator algebra in the frame of stochastic mechanics.
Phys. Rev. D, 28(8) :1916–1921.

Références II

Littlejohn, R. B. (2021).
Notes 22 - Time Reversal.
In *Physics 221A Academic Year 2021-22*. Berkeley.

Schrödinger, E. (1931).
Über die Umkehrung der Naturgesetze. (German) [On the reversal of the laws of nature].
j-S-B-PREUSS-AKAD-WISS-PHYS-MATH-KL, pages 144–153.

Schrödinger, E. (1932).
Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique.
Ann. Inst. Henri Poincaré, 2(4) :43.

Références III

Zambrini, J. C. (1986).
Stochastic mechanics according to E. Schrödinger.
Phys. Rev. A, 33(3) :1532–1548.

Zambrini, J. C. (1987).
Euclidean quantum mechanics.
Phys. Rev. A, 35(9) :3631–3649.

Starting Theory	Classical Mechanics	Classical Mechanics
Symmetry invariance	INERTIAL FRAME velocity transformations $v' = v + u$	Scaling transformations of ACTION $S' = e^{-\alpha} S$
Universal requirement	$v \leq c$	$Var(x) \cdot Var(p) \geq \frac{\hbar^2}{4}$
Conceptual changes to restore invariance	Momentum: $p = mv$ ⇓ $\tilde{p} = \frac{mv}{\sqrt{1-(\frac{v}{c})^2}}$ Energy: $E = \frac{mv^2}{2}$ ⇓ $\tilde{E} = \sqrt{\tilde{p}^2 c^2 + m^2 c^4}$	Momentum Uncertainty: $Var(p) = \int d^3x \, \rho \, \lvert \vec{\nabla} S \rvert^2$ ⇓ $\Delta_p^2 = Var(p) + \frac{\hbar^2}{4} \frac{1}{\Delta_x^2}$ with: $\Delta_x^2 = \frac{1}{4 \int d^3x \, \lvert \vec{\nabla} \rho^{1/2} \rvert^2}$ Energy: $E = Var(p)/2m$ ⇓ $\tilde{E} = Var(p)/2m + \frac{\hbar^2}{8m} \frac{1}{\Delta_x^2}$ $= \int d^3x \, \psi^*(x) \left(-\frac{\hbar^2}{2m} \nabla^2\right) \psi(x)$.
Consequences on geometry	Velocity transformation couple Space and time $ct' = \cosh\beta \cdot ct - \sinh\beta \cdot x$ $x' = -\sinh\beta \cdot ct + \cosh\beta \cdot x$	Scaling transformations of action couple time and a new time-like parameter $t' = e^{\alpha}[\cosh\alpha \cdot t + \sinh\alpha \cdot \tau]$ $\tau' = e^{\alpha}[\sinh\alpha \cdot t + \cosh\alpha \cdot \tau]$ Evolution in τ is generated by $K = Var(p)/2m - \frac{\hbar^2}{8m} \frac{1}{\Delta_x^2}$
Resulting theory	SPECIAL RELATIVITY	Quantum Mechanics + Schrödinger Bridge process